Journal of Engineering Mathematics **37:** 397–400, 2000.

Short Communication

Note on a paper by Sinha and Odgaard "Application of conformal mapping to diverging open channel flow"

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Received 11 June 1998; accepted in revised form 18 November 1999

Abstract. A steady gravity-free two-dimensional potential flow separating a parent channel and an offtake channel is studied by conformal mapping of a half-strip in the Zhukovskii domain onto a strip with a cut in the complex potential domain. A parameter of the mapping and the equation of a free streamline are found by computer-algebra routines.

Keywords: complex potential, free surface, potential flow, channel

1. Introduction

Free-surface flows are thoroughly studied by the method of complex variables in many branches of fluid mechanics, hydrology, and subsurface hydrology [1,2]. The isobaric condition along free boundaries allows implementing the Zhukovskii function or hodograph. Mapping them onto the corresponding domain in the complex potential plane through an auxiliary plane reconstructs the flow pattern. Recent applications of conformal mappings to free-surface flows are presented in [3–8]. Sinha and Odgaard [9] studied bifurcating channel flows, which occur in practice (*e.g.* [10] pp. 512–516, [11], pp. 183–234). The complex potential domain used in [9] for mapping is a strip. The errors of this approximation are discussed in [8] where the hodograph method has been used. In this note, we derive a solution to the problem from [9] assuming a strip with a cut in the complex-potential domain and calculate the free streamline in a way different from [8].

2. Analysis

We consider the flow pattern from [9]. Figure 1a shows the physical plane $z = x + \mathrm{i}y$ and Figure 1b shows the Zhukovskii function plane $Q = \log(V_2/V) + \mathrm{i}\theta$. We conserve all notations from [9]. In particular, we assume that D is a separation point, *i.e.* the flow is balanced in terms of [8]. The complex-potential ($w = \Phi + \mathrm{i}\Psi$) domain used in [9, Figure 4] does not correspond to the flow pattern with a separatrice DA in our Figure 1a and, hence, we modify this domain into a strip with a cut (Figure 1c). Correspondingly, we modify the Equations (24)–(25), (27)–(28) from [9]. We map conformally the half-strip of the Q-plane onto the upper-half of the auxiliary plane $\lambda = \xi + \mathrm{i}\nu$ (Figure 1d) by:

$$Q = -\frac{\mathrm{i}\alpha}{\pi} \arcsin\lambda + \mathrm{i}(\pi - \alpha/2), \tag{1}$$

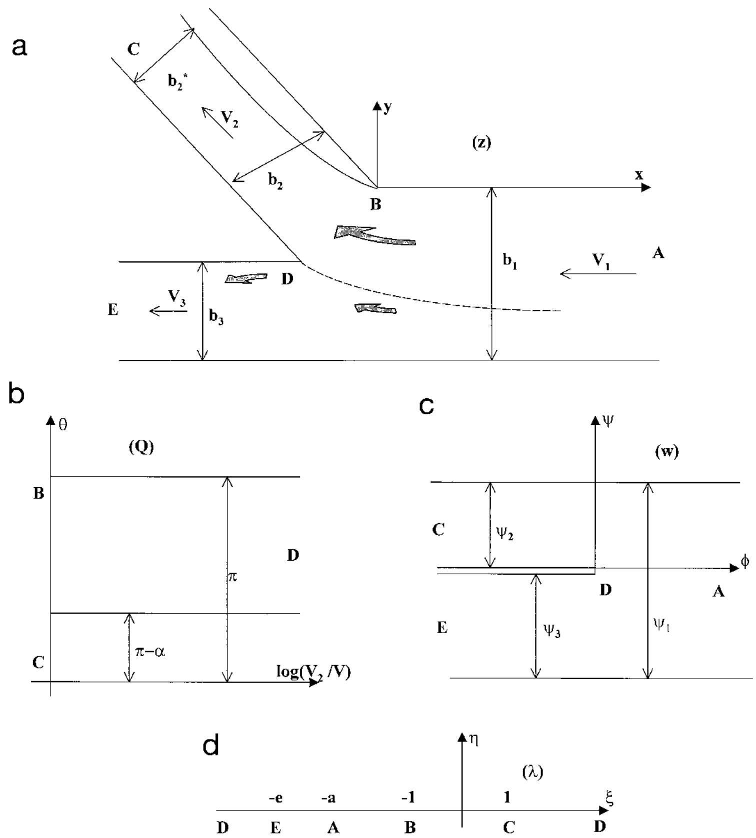

Figure 1. Physical plane (a), Zhukovskii plane (b), complex potential plane (c), auxiliary half-plane (d).

This equation is analogous to Equation (21) from [9]. Obviously, at $\lambda = \xi > 1$ (1) yields:

$$\log(V_2/V) = \frac{\alpha}{\pi}\log(\xi + \sqrt{\xi^2 - 1}). \tag{2}$$

From (2)

$$\frac{V_2}{V_1} = (a + \sqrt{a^2 - 1})^{\alpha/\pi}, \tag{3}$$

$$\frac{V_2}{V_3} = (e + \sqrt{e^2 - 1})^{\alpha/\pi}. \tag{4}$$

We assume that b_1, b_2, b_3, V_1, V_2 are known. The downstream velocity in the parent channel V_3 and the asymptotic flow width in the offtake channel b_2^* have to be determined. From (3) a is immediately expressed as:

$$a = \frac{p^2 + 1}{2p}, \quad \text{where} \quad p = \left(\frac{V_2}{V_1}\right)^{\pi/\alpha}. \tag{5}$$

Conformal mapping of the complex potential plane onto λ yields:

$$w = A\left[\frac{\log(\lambda - 1)}{(a+1)(e+1)} + \frac{\log(\lambda + a)}{(a+1)(a-e)} - \frac{\log(\lambda + e)}{(e+1)(a-e)}\right]. \tag{6}$$

Our mapping function (6) is just the same as for the case of flow without a free streamline ([9] formula (15)). The four unknown values e, A, V_3, b_2^* are connected through the three jump conditions at the points E, A, B as:

$$\frac{A\pi}{(a+1)(e-a)} = \Psi_1 = b_1 V_1, \tag{7}$$

$$\frac{A\pi}{(a+1)(e+1)} = \Psi_2 = b_2^* V_2, \tag{8}$$

$$b_1 V_1 = b_2^* V_2 + b_3 V_3. \tag{9}$$

To close the system we have to determine b_2^*. For this purpose, we express the z-function as:

$$z(\lambda) = \frac{A}{V_2} \int_{-1}^{\lambda} \frac{\mathrm{e}^{\Omega}(\tau)\,\mathrm{d}\tau}{(\tau-1)(\tau+a)(\tau+e)}. \tag{10}$$

From Equation (10) at $\lambda = \xi$, $-1 \le \xi \le 1$, the parametric equations for the free streamline BC are

$$x(\xi) = \frac{A}{V_2} \int_{-1}^{\xi} \frac{\cos(\alpha/\pi \ \arcsin\,\tau + \alpha/2)\,\mathrm{d}\tau}{(\tau-1)(\tau+a)(\tau+e)}, \tag{11}$$

$$y(\xi) = -\frac{A}{V_2} \int_{-1}^{\xi} \frac{\sin(\alpha/\pi \ \arcsin\,\tau + \alpha/2)\,\mathrm{d}\tau}{(\tau-1)(\tau+a)(\tau+e)}. \tag{12}$$

From Equations (11)–(12) $b_2^* = b_2 + x(1)\,\sin\,\alpha + y(1)\,\cos\,\alpha$. We put this value, the value of A expressed from Equation (7), and V_3 expressed through V_2 and e from Equation (4) into Equation (9). As a result we come to an equation for e:

$$\frac{V_1}{V_2} = \frac{b_2}{b_1} + \frac{V_1}{V_2}\frac{(e-a)(e+1)}{\pi} \times \int_{-1}^{1} \frac{\sin(\alpha/2 - \alpha/\pi \ \arcsin\,\tau)\,\mathrm{d}\tau}{(\tau-1)(\tau+a)(\tau+e)} + \frac{b_3}{b_1(e+\sqrt{e^2-1})^{\alpha/\pi}} \tag{13}$$

We solved this equation by standard methods [12]. We calculated V_3, b_2^*, and the free streamline according to Equations (11)–(12). Figure 2 illustrates the free streamlines plotted for $b_3/b_1 = 1$, $b_2/b_1 = 0.1$, $\alpha = \pi/2$ and $V_2/V_1 = 2, 7, 12$ with corresponding values $V_3/V_1 = 0{\cdot}92, 0{\cdot}65, 0{\cdot}36$ (curves 1–3, respectively).

3. Conclusions

The Schwarz-Christoffel formula was used for conformal mapping of one characteristic domain (complex potential) onto the hodograph or Zhukovskii function domain in many problems of fluid mechanics (*e.g.* [2], pp. 107–109, 231–239). In this note, we apply this formula to a problem when a uniform flow in a straight parent channel splits between the narrowed or widened continuation of the parent channel and an offtake channel such that in the offtake channel a free (isobaric) streamline appears. Unlike [9], we consider the case when the velocity potential decreases after bifurcation from $+\infty$ in the upstream part of the parent channel

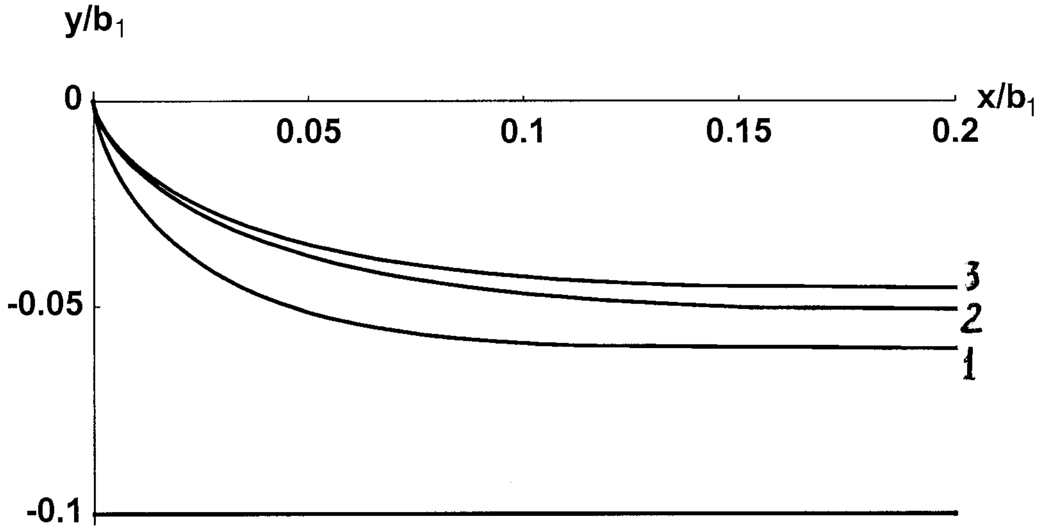

Figure 2. Free stream-lines for $b_3/b_1 = 1$, $b_2/b_1 = 0.1$, $\alpha = \pi/2$ and $V_2/V_1 = 2$, 7, 12 with corresponding values $V_3/V_1 = 0{\cdot}92$, $0{\cdot}65$, $0{\cdot}36$ (curves 1–3, respectively).

(far from the bisection point) to $-\infty$ both in the downstream branch of the parent channel and in the offtake channel. Hence, we modify the solution from [9] , taking into account a cut in the complex potential domain. This cut corresponds to the wall, which intersects the bifurcating separatrice. The free streamline is found after solution of a nonlinear equation with respect to one parameter of the mapping. Unlike [8] the free streamline is calculated by use of integral representations similar to [9].

Acknowledgements

Helpful comments of three anonymous referees are appreciated, especially, their pointing out the paper by Hassenpflug [8], which was published after our work was done and submitted.

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